

ON ARITHMETIC PROGRESSIONS IN RECURRENCES - A NEW CHARACTERIZATION OF THE FIBONACCI SEQUENCE

ÁKOS PINTÉR AND VOLKER ZIEGLER

ABSTRACT. We show that essentially the Fibonacci sequence is the unique binary recurrence which contains infinitely many three-term arithmetic progressions. A criterion for general linear recurrences having infinitely many three-term arithmetic progressions is also given.

1. INTRODUCTION

Linear recurrence sequences have been studied since ancient times and also in the last century the interest in recurrences was high. Equations concerning linear recurrences have an extremely rich literature. For instance, let f_n be a recurrence sequence, then the equation $f_n = 0$ has been studied by several mathematicians. The finiteness of zero-multiplicity of f_n was proved by Skolem, Mahler and Lech [22, 10, 8]. Although one can give upper bounds for the number of solutions (see [19]) in general, it is not possible to find all solutions effectively. However, in the case of binary and ternary recursions Mignotte [11] found effective growth estimates and therefore in these cases we can give all values of n with $f_n = 0$. But also equations of the type $Af_n = Bg_m$ were studied by various authors (e.g. Schlickewei and Schmidt [18]). For a full account to study similar linear equations in recurrence sequences we recommend the book of Everest et. al. [5].

Let K be an algebraically closed field of characteristic zero, Γ a multiplicative subgroup of K^* having finite rank r , let \mathcal{A} be a finite set of t -tuples $\in K^t$ having n elements and put

$$H_t(\Gamma, \mathcal{A}) = \left\{ \sum_{i=1}^t a_i x_i : (a_1, \dots, a_t) \in \mathcal{A}, (x_1, \dots, x_t) \in \Gamma^t \right\}.$$

Hajdu [6] proved that there exists a constant $C(r, t, n)$ such that there exists no non-constant arithmetic progression in $H_t(\Gamma, \mathcal{A})$ with length $\geq C(r, t, n)$. A direct consequence of this result is that the length of arithmetic progressions in simple recurrence sequences is bounded by a constant depending only on its order d . Recently, the interest in arithmetic progressions in certain number-theoretical structures, like the points on elliptic curves [3], solutions of Pellian equations [12, 4] or norm form equations [2], has increased. We also mention here a new result due to by Schwartz, Solymosi and de Zeeuw [20].

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The purpose of this paper is to connect to these investigations. Roughly speaking we show that sequences that contain infinitely many three-term arithmetic progressions are very special. Note that finding non-trivial three term arithmetic progressions (f_m, f_n, f_k) is equivalent to solve the equation

$$(1) \quad f_m + f_k = 2f_n.$$

Equations of the type $Af_m + Bf_n + Cf_k = 0$ have been studied by Schlickewei and Schmidt [17]. Before stating our results we introduce some notions.

A linear recurrence f_n of order d is a complex sequence satisfying the recurrence

$$f_{n+d} = a_{d-1}f_{n+d-1} + \cdots + a_0f_n$$

with $a_i \in \mathbb{C}$ for $i = 0, \dots, d-1$, $a_0 \neq 0$ and the sequence does not satisfy such an equation with fewer summands. The companion polynomial P is defined by

$$P(X) = X^d - a_{d-1}X^{d-1} - \cdots - a_0.$$

A linear recurrence is simple if its companion polynomial P has simple zeros only, and it is called non-degenerate if α_i/α_j is not a root of unity for any distinct zeros α_i and α_j of P . Further, a recurrence is called a unitary sequence if its companion polynomial possesses at least one zero which is a root of unity.

Let $\alpha_1, \dots, \alpha_r$ be the zeros of the companion polynomial P and assume that α_i is a zero of multiplicity σ_i . Then we can write

$$f_n = \sum_{i=1}^r p_i(n) \alpha_i^n,$$

where $p_i(n)$ are polynomials of degree $< \sigma_i$.

Using the above cited result by Schlickewei and Schmidt we prove the following

Theorem 1. *Let f_n be a non-degenerate and non-unitary recurrence with companion polynomial P . Then there is a finite set $S_0 \subset \mathbb{N}^3$ such that all three-term arithmetic progressions (f_m, f_n, f_k) with $f_n \neq 0$ satisfy $(m, n, k) \in S_0$ (isolated solutions) or one of the following three cases occurs:*

- *All but finitely many solutions to (1) are of the form $m = k + a, n = k + b$, with $a, b \in \mathbb{Z}$ and $P(X)|(X^a - 2X^b + 1)X^{-\min\{a, b, 0\}}$.*
- *The recursion is of the form*

$$(2) \quad f_n = \sum_{i=1}^r c_i \left(\alpha_{2i-1}^n + \alpha_{2i}^n \frac{\alpha_{2i-1}^{a+c} + \alpha_{2i-1}^{b+c}}{2} \zeta_i^c \right), \text{ with}$$

$$0 = (\zeta_i^a + \zeta_i^b - 4\zeta_i^c) + \zeta_i^a \alpha_j^{b-a} + \zeta_i^b \alpha_j^{a-b} \text{ or}$$

$$(3) \quad f_n = \sum_{i=1}^r c_i (\alpha_{2i-1}^n + \alpha_{2i}^n (\alpha_{2i-1}^{a+c} + 2\alpha_{2i-1}^{b+c}) \zeta_i^c), \text{ with}$$

$$0 = (\zeta_i^a + 4\zeta_i^b - \zeta_i^c) - 2\zeta_i^a \alpha_j^{b-a} - 2\zeta_i^b \alpha_j^{a-b} \text{ or}$$

$$(4) \quad f_n = \sum_{i=1}^r c_i (\alpha_{2i-1}^n + \alpha_{2i}^n (2\alpha_{2i-1}^{a+c} + \alpha_{2i-1}^{b+c}) \zeta_i^c), \text{ with}$$

$$0 = (4\zeta_i^a + \zeta_i^b - \zeta_i^c) - 2\zeta_i^a \alpha_j^{b-a} - 2\zeta_i^b \alpha_j^{a-b},$$

where $j = 2i-1, 2i$, $c_i \in \mathbb{C}$, $\alpha_{2i-1}\alpha_{2i} = \zeta_i$ is an M -th root of unity with M minimal for all $i = 1, \dots, r/2$. Then according to (2), (3) or (4) (f_m, f_k, f_n)

or (f_k, f_n, f_m) or (f_n, f_m, f_k) with $m = Mt + a, n = Mt + b, k = -Mt + c$ are arithmetic progressions for all integers t .

- The recursion is of the form

$$f_n = C(n - \gamma)2^{n/K} \zeta_K^n$$

where ζ_K is a K -th root of unity, with $\gamma, K \in \mathbb{Z}$ and $C \in \mathbb{C}$. Then f_n, f_m and f_k form an arithmetic progression (arranged in some order) if $n = c2^s + \gamma, m = c2^s + as + b, k = c2^s + a's + b'$ with a, a', b, b', c integers for all integers $s \geq 0$. Moreover K and c cannot be both positive.

We exclude the case $(f_m, 0, f_k)$ since this leads to the equation $f_m = -f_k$ which is not an essential restriction for so called symmetric recurrences. In order to keep Theorem 1 as short as possible (which is not an easy task) we made this technical restriction. Note that the other cases are essential restrictions for the recurrences. Therefore excluding this case recurrences which admit infinitely many three-term arithmetic progressions are in some way very special. How special they are can be seen in Corollary 2. However, it is no problem to include conditions under which $(f_m, 0, f_k)$ is an arithmetic progression.

Also, remark that we can bound the number of isolated solutions $|S_0|$ but we cannot give an upper estimate for the “maximum” of these solutions. The reason lies in the use of the quantitative version of the subspace theorem. We want to point out here that in many important cases we can compute S_0 effectively. At least this can be done for all binary and ternary recurrences (for techniques to do see [11]).

If we restrict ourselves to recurrences defined over the integers, i.e. $f_n \in \mathbb{Z}$ for all $n \in \mathbb{Z}$, and consider only positive indices we obtain.

Corollary 1. *Let f_n be non-degenerate, non-unitary and be defined over the integers. Moreover, assume f_n provides infinitely many three-term arithmetic progressions (f_m, f_n, f_k) with $n, m, k > 0$. Then the companion polynomial $P(X)$ is one of the factors of $\frac{X^a - 2X^b + 1}{X^d - 1}$ with $a > b > 0$ and $d = \gcd(a, b)$.*

Note that the factorization of trinomials has been extensively studied by Schinzel (see e.g. his book [16]). In particular the precise factorization of the polynomial $X^a - 2X^b + 1$ for $a > b > 0$ is known (see [13]). Schinzel used the factorization of $X^a - 2X^b + 1$ to prove that there exist no non-trivial four-term arithmetic progression in sequences of the form $f_n = q^n$ and q an irrational number (a question due to Sierpiński [21]). In Lemma 3 we will give the factorization of $X^a + X^b - 2$. These results on the factorization of trinomials are crucial in the proof of the following theorem on the binary recurrence case.

Theorem 2. *Let f_n be a non-degenerate and non-unitary binary recurrence, which is defined over the rationals and contains infinitely many three-term arithmetic progressions. Then f_n fulfills one of the following conditions*

- The binary recurrence f_n is of the form $f_n = R(n - \gamma)2^{\pm n}$, with $R \in \mathbb{Q}^*$ and $\gamma \in \mathbb{Z}$. Such recurrences admit arithmetic three term progressions (f_m, f_n, f_k) with

$$m = \mp 2^{s \mp \gamma} \pm s, \quad n = \mp 2^{s \mp \gamma} \pm s \mp 1, \quad k = \mp 2^{s \mp \gamma} + \gamma$$

for all $s > \pm \gamma + \frac{1 \pm 1}{2}$.

- The sequence is listed in Table 1 (up to a multiplication by a rational) and f_m, f_n and f_k form a three-term arithmetic progression (in some order) with $m = 2t + a, n = 2t + b$ and $k = -2t + c$ for all $t \in \mathbb{Z}$.
- The companion polynomial of the recurrence f_n is listed in Table 2.

TABLE 1. Sequences that contain infinitely many arithmetic progressions involving f_{2t+a}, f_{2t+b} and f_{-2t+c}

f_n	α	a, b, c
$f_n = C \left(\alpha^n + (-1)^{n+c} \alpha^{b+c-n} \frac{1+\alpha}{2} \right)$ $C = 1 + \frac{(-1)^a \alpha^{-(a+c)} + (-1)^b \alpha^{-(b+c)}}{2}$	$\alpha = 2 \pm \sqrt{5}$	$a - b = 1$ $b + c \equiv 0 \pmod{2}$
	$\alpha = -2 \pm \sqrt{5}$	$a - b = 1$ $b + c \equiv 1 \pmod{2}$
$f_n = C \left(\alpha^n + (-1)^{n+c} \alpha^{b+c-n} \frac{1+\alpha^3}{2} \right)$ $C = 1 + \frac{(-1)^a \alpha^{-(a+c)} + (-1)^b \alpha^{-(b+c)}}{2}$	$\alpha = \frac{1 \pm \sqrt{5}}{2}$	$a - b = 3$ $b + c \equiv 0 \pmod{2}$
	$\alpha = \frac{-1 \pm \sqrt{5}}{2}$	$a - b = 3$ $b + c \equiv 1 \pmod{2}$
$f_n = C \left(\alpha^n + (-1)^{n+c+1} \alpha^{b+c-n} (2\alpha - 1) \right)$ $C = 1 + (-1)^b \alpha^{-(b+c)} - (-1)^a 2\alpha^{-(a+c)}$	$\alpha = -1 - \sqrt{2}$	$a - b = 1$ $b + c \equiv 0 \pmod{2}$
	$\alpha = -\frac{1+\sqrt{5}}{2}$	$a - b = 1$ $b + c \equiv 1 \pmod{2}$
$f_n = C \left(\alpha^n + (-1)^{n+c} \alpha^{b+c-n} (\alpha - 2) \right)$ $C = 1 + (-1)^a \alpha^{-(a+c)} - (-1)^b 2\alpha^{-(b+c)}$	$\alpha = -1 - \sqrt{2}$	$a - b = 1$ $b + c \equiv 1 \pmod{2}$
	$\alpha = -\frac{1+\sqrt{5}}{2}$	$a - b = 1$ $b + c \equiv 0 \pmod{2}$

TABLE 2. Companion polynomials of binary recursion containing arithmetic progressions with f_{n+a}, f_{n+b} and f_n involved.

a	b	$P(X)$
3	1	$X^2 + X - 1$
		$X^2 + X + 2$
		$2X^2 + 2X + 1$
3	2	$X^2 - X - 1$
		$X^2 + 2X + 2$
		$2X^2 + X + 1$

Let us consider the Fibonacci sequence, i.e.

$$f_0 = 0, \quad f_1 = 1, \quad f_{n+2} = f_{n+1} + f_n, \quad n \geq 0.$$

We obtain the following characterization of the Fibonacci sequence.

Corollary 2. *The only increasing, simple, non-degenerate and non-unitary recursion f_n defined over the rationals that contains infinitely many three-term arithmetic progressions (f_m, f_n, f_k) with $m, n, k \geq 0$, which additionally satisfies $f_0 = 0$ and $f_1 = 1$ is the Fibonacci sequence.*

Moreover, the Fibonacci sequence contains for $n \geq 0$ beside the infinite family (f_n, f_{n+2}, f_{n+3}) of three-term arithmetic progressions only the three-term arithmetic progressions

$$(f_0 = 0, f_1 = 1, f_3 = 2), (f_0 = 0, f_2 = 1, f_3 = 2) \text{ and } (f_2 = 1, f_3 = 2, f_4 = 3).$$

The only four-term arithmetic progressions are

$$(f_0 = 0, f_1 = 1, f_3 = 2, f_4 = 3) \text{ and } (f_0 = 0, f_2 = 1, f_3 = 2, f_4 = 3).$$

The condition non-unitary is essentially since the sequence $f_n = \frac{2^n - (-1)^n}{3}$ fulfills the same properties as required in the corollary and contain the infinite family of arithmetic three-term progressions $(f_{2t}, f_{2t-1}, f_1 = f_2)$. However, by simple growth estimates we can show that this sequence is the only exception.

Corollary 3. *Omitting the condition non-unitary in Corollary 2, we have $f_n = \frac{2^n - (-1)^n}{3}$ or f_n is the Fibonacci sequence.*

Although Theorem 2 is long and technical the case of ternary sequences is much easier, since the so-called symmetric and exceptional cases do not occur. Therefore we show

Theorem 3. *Let f_n be a non-degenerate, non-unitary, ternary recurrence, which is defined over the rationals and contains infinitely many arithmetic progressions. Then f_n has companion polynomial listed in Table 3.*

TABLE 3. Companion polynomials of ternary recursion containing arithmetic progressions with f_{n+a} , f_{n+b} and f_n involved.

a	b	$P(X)$
4	1	$X^3 + X^2 + X - 1$
		$X^3 + X^2 + X + 2$
		$2X^3 + 2X^2 + 2X + 1$
4	3	$X^3 - X^2 - X - 1$
		$X^3 + 2X^2 + 2X + 2$
		$2X^3 + X^2 + X + 1$
7	2	$X^3 + X^2 + 1$
		$X^3 - X - 1$
7	5	$X^3 + X^2 - 1$
		$X^3 + X - 1$

2. NOTATION AND LINEAR EQUATIONS IN RECURRENCES

We start this section with some notions. In the sequel we assume that f_n is a non-degenerate and non-unitary linear recurrence sequence with companion polynomial P . Let $\alpha_1, \dots, \alpha_r$ be the zeros of P . We call f_n symmetric if r is even and the zeros $\alpha_1, \dots, \alpha_r$ can be arranged such that $(\alpha_i \alpha_{i+1})^M = 1$ for each odd $1 \leq i < r$. We call f_n exceptional if there exists an integer $N > 0$ such that each α_i is a rational power of N , each $|\alpha_i| > 1$ or each $|\alpha_i| < 1$ and $p_i(n) = \gamma_i(n - \gamma)$ with $\gamma \in \mathbb{Q}$. Note that a recurrence cannot be both symmetric and exceptional. We are interested in the two equations

$$(5) \quad Af_n = Bf_m$$

and

$$(6) \quad Af_n + Bf_m + Cf_k = 0, \quad f_n f_m f_k \neq 0$$

where $ABC \neq 0$. These equations were investigated by Laurent [7] and Schlickewei and Schmidt [17], respectively. The next three Propositions are reformulations of [17, Proposition 1 and 2, Theorem 1 and 2].

Let us consider the case where f_n is neither symmetric nor exceptional. Then we have

Proposition 1. *Let A, B, C be non-zero constants and let f_n be neither symmetric nor exceptional. Then all solutions to (5) but finitely many are contained in the one parameter family $n = t + a$ and $m = t + b$ for certain $a, b \in \mathbb{Z}$. Moreover all but finitely many solutions satisfy*

$$(7) \quad Ap_i(n)\alpha_i^n = Bp_i(m)\alpha_i^m.$$

All solutions to the ternary equation (6) but finitely many are contained in one of finitely many families of the form

$$(8) \quad \mathcal{F}_j : \quad n = k + a_j, \quad m = k + b_j, \quad a_j, b_j \in \mathbb{Z}$$

and satisfy the polynomial identity

$$(9) \quad Ap_i(n)\alpha_i^n + Bp_i(m)\alpha_i^m + Cp_i(k)\alpha_i^k = 0.$$

Now we consider the symmetric case. Assume we have arranged the roots as described above. In this case further solutions may occur:

Proposition 2. *Let A, B, C be non-zero constants and let f_n be symmetric. Then the equation (5) has the additional family of solutions $n = Mt + a'$ and $m = -Mt + b'$ for certain $a', b' \in \mathbb{Z}$. These solutions satisfy the system*

$$(10) \quad \begin{aligned} Ap_i(n)\alpha_i^n &= Bp_{i+1}(m)\alpha_{i+1}^m, \\ Ap_{i+1}(n)\alpha_{i+1}^n &= Bp_i(m)\alpha_i^m, \end{aligned}$$

for all odd i with $1 \leq i \leq r$. Solutions to the ternary equation (6) may lie in one of the additional families of solutions $\mathcal{S}_j^{(n)}, \mathcal{S}_j^{(m)}$ or $\mathcal{S}_j^{(k)}$, where e.g.

$$(11) \quad \mathcal{S}_j^{(k)} : \quad n = Mt + a_j^{(k)}, \quad m = Mt + b_j^{(k)} \quad k = -Mt + c_j^{(k)};$$

where the a 's, b 's and c 's are integers. All additional solutions satisfy a corresponding system of equations, e.g. for the family $\mathcal{S}_j^{(k)}$ we have

$$(12) \quad \begin{aligned} Ap_i(n)\alpha_i^n + Bp_i(m)\alpha_i^m + Cp_{i+1}(k)\alpha_{i+1}^k &= 0, \\ Ap_{i+1}(n)\alpha_{i+1}^n + Bp_{i+1}(m)\alpha_{i+1}^m + Cp_i(k)\alpha_i^k &= 0, \end{aligned}$$

for all odd i with $1 \leq i \leq r$. The other equations are obtained by permuting indices.

Finally, in the exceptional case we obtain

Proposition 3. *Equation (5) has no additional solutions in view of Proposition 1. All but finitely many solutions to the ternary equation (6) satisfy (9). But additional solutions may lie in one of the finitely many exceptional families $\mathcal{E}_j^{(n)}, \mathcal{E}_j^{(m)}$ or $\mathcal{E}_j^{(k)}$, where e.g.*

$$(13) \quad \mathcal{E}_j^{(n)} : \quad n = c_j N^s + \gamma, \quad m = c_j N^s + as + b_j \quad k = c_j N^s + a's + b'_j.$$

These additional solutions appear only if all $p_i(n) = \gamma_i(n - \gamma)$. Further $c_j \in \mathbb{Q}^$ and $a, a', b_j, b'_j \in \mathbb{Q}$ are such that $(n(s), m(s), k(s)) \in \mathbb{Z}^3$ for each $s \in \mathbb{Z}$, $s \geq 0$.*

3. PROOF OF THEOREM 1

For the proof of Theorem 1 we have to consider equation (6) with one of A, B, C is equal to -2 and the other coefficients are equal to 1. The case where $f_m f_n f_k = 0$ has to be considered separately. We divide the proof of Theorem 1 into the obvious three subcases, i.e. f_n is symmetric, exceptional or neither of them. Let us start with the case where f_n is neither symmetric nor exceptional.

3.1. The general case. First we assume $f_m f_n f_k \neq 0$. Then we may assume that all but finitely many solutions are of the form $m = k + a, n = k + b$ with $a > b > 0$ and they satisfy the equation

$$Ap_i(k+a)\alpha_i^a + Bp_i(k+b)\alpha_i^b + Cp_i(k) = 0$$

for all $1 \leq i \leq r$ (see Proposition 1). Fix the index i and write for simplicity $\alpha = \alpha_i$ and $p_i(k) = p(k) = A_d k^d + \dots + A_0$. Considering the equation above for $k \rightarrow \infty$ we see that the equation has to be satisfied polynomial. Assume $d > 0$. We compare the coefficients of k^d in the equation and find

$$(14) \quad AA_d \alpha^a + BA_d \alpha^b + CA_d = 0$$

and for k^{d-1} we find

$$(15) \quad A\alpha^a(A_d da + A_{d-1}) + B(A_d db + A_{d-1}) + CA_{d-1} = 0$$

Subtracting equation (14) from (15) and after some calculations we obtain the system

$$\begin{aligned} A\alpha^a + B\alpha^b &= -C \\ Aa\alpha^a + Bb\alpha^b &= 0 \end{aligned}$$

Solving for α^a and α^b yields $\alpha^a = -bBC$ and $\alpha^b = aAC$. Assume $A = -2$ then we have $\alpha^a = -b$ and $\alpha^b = -2a$. Taking the first relation to the b -th power and inserting the second we obtain $(-b)^b = (-2a)^a$. In the case of $B = -2$ or $C = -2$ we obtain $(2b)^b = a^a$ or $(2b)^b = (-2a)^a$, respectively. The last equation has obviously no integral solution with $a > b > 0$. The other two cases have also no solution because of the next lemma.

Lemma 1. *The equation $a^a = (2b)^b$ has no positive integral solution.*

Proof. First, note that the equation implies $a > b$, i.e. $a = xb$ with $x > 1$, $x = p/q$ and $p, q \in \mathbb{Z}$ with $\gcd(p, q) = 1$. Inserting for $a = xb$ the equation is equivalent to

$$b^{x-1}x^x = 2$$

taking q -th powers we have

$$b^{p-q}x^p = 2^q$$

a rational equation. Let $r \neq 2$ be a prime dividing p . Computing the r -adic valuations on the left and right hand side we obtain

$$\beta_r(p-q) + p\alpha_r = 0,$$

where β_r and α_r are the r -adic valuations of b and p . Since $\alpha_r > 0$, $\beta_r \geq 0$ and $p > q$ we have a contradiction. Therefore $p = 2^k$ and we consider 2-adic valuations:

$$\beta_2(p-q) + pk = q.$$

Since $p > q$ we obtain again a contradiction unless $k = 0$. Hence $x = 1/q \leq 1$ again a contradiction, i.e. the equation has no solution. \square

Therefore all p_i are constant and all α have to satisfy either of the equations

$$-2X^a + X^b + 1, \quad X^a - 2X^b + 1, \quad X^a + X^b - 2,$$

with $a > b > 0$.

Now we consider the case $f_m f_n f_k = 0$. Since we excluded the case $(f_m, 0, f_k)$ we are lead to the equation $2f_n = f_k$. By Proposition 1 we have

$$(16) \quad 2p_i(k+a)\alpha_i^{k+a} = p_i(k)\alpha_i^k.$$

Dividing through α_i^k and then taking the limit $k \rightarrow \infty$ we obtain $\alpha_i^a = 1/2$ for all $1 \leq i \leq r$. Since the recurrence f_n is non-degenerate we find $r = 1$ and $\alpha_1 = \alpha = 2^{-1/a}\zeta_a$ where ζ_a is some a -th root of unity. If we insert this into (16) we obtain $p(k+a) = p(k)$ which on the other hand tells us $p(k)$ is constant. Therefore $f_n = c2^{-n/a}$. On the other hand we have $f_m = 0$ hence $c = 0$. Therefore f_n is a constant recurrence which we excluded.

3.2. The symmetric case. Now, let us treat the symmetric case. Let us write $\alpha_i \alpha_{i+1} = \zeta_i$ with $\zeta_i^M = 1$ and as the first case let us assume $f_m f_n f_k \neq 0$. In this case all solutions but finitely many lie in one of the families \mathcal{F}_j or in $\mathcal{S}_j^{(m)}, \mathcal{S}_j^{(n)}$ or $\mathcal{S}_j^{(k)}$. The case where the solution lies in \mathcal{F}_j is identical with the case treated in the subsection above. Therefore we may assume $m = Mt + a, n = Mt + b$ and $k = -Mt + c$. According to Proposition 2 we have to distinguish three cases. For each odd i we have

$$(17) \quad \begin{aligned} & Ap_i(Mt+a)\alpha_i^{Mt+a} + Bp_i(Mt+b)\alpha_i^{Mt+b} + Cp_{i+1}(-Mt+c)\alpha_{i+1}^{-Mt+c} = 0; \\ & Ap_{i+1}(Mt+a)\alpha_{i+1}^{Mt+a} + Bp_{i+1}(Mt+b)\alpha_{i+1}^{Mt+b} + Cp_i(-Mt+c)\alpha_i^{-Mt+c} = 0 \end{aligned}$$

or an equation which is obtained from (17) by permuting A, B and C .

Let us multiply the first equation by α_i^{-Mt} and the second by α_{i+1}^{-Mt} . Then we get

$$\begin{aligned} & Ap_i(Mt+a)\alpha_i^a + Bp_i(Mt+b)\alpha_i^b + Cp_{i+1}(-Mt+c)\alpha_{i+1}^c = 0; \\ & Ap_{i+1}(Mt+a)\alpha_{i+1}^a + Bp_{i+1}(Mt+b)\alpha_{i+1}^b + Cp_i(-Mt+c)\alpha_i^c = 0. \end{aligned}$$

We note that the polynomials p_i and p_{i+1} are of the same degree, because otherwise either the first equation or the second would yield a contradiction if we divide through p_i and t tends to infinity. Now let us assume that p_i and p_{i+1} are of degree $d \geq 1$ and assume

$$\begin{aligned} p_i(t) &= A_d^{(i)}t^d + A_{d-1}^{(i)}t^{d-1} + \dots; \\ p_{i+1}(t) &= A_d^{(i+1)}t^d + A_{d-1}^{(i+1)}t^{d-1} + \dots \end{aligned}$$

and write $q_i = A_d^{(i+1)}/A_d^{(i)}$. Comparing coefficients of t^d and t^{d-1} yields the system of equations

$$\begin{aligned} AA_d^{(i)} M^d \alpha_i^a + BA_d^{(i)} M^d \alpha_i^b + CA_d^{(i+1)} M^d \alpha_{i+1}^c &= 0; \\ AA_d^{(i+1)} M^d \alpha_{i+1}^a + BA_d^{(i+1)} M^d \alpha_{i+1}^b + CA_d^{(i)} M^d \alpha_i^c &= 0; \\ M^{d-1} A \alpha_i^a (A_d^{(i)} ad + A_{d-1}^{(i)}) + M^{d-1} B \alpha_i^b (A_d^{(i)} bd + A_{d-1}^{(i)}) \\ &\quad + M^{d-1} C \alpha_{i+1}^c (A_d^{(i+1)} cd + A_{d-1}^{(i+1)}) = 0; \\ M^{d-1} A \alpha_{i+1}^a (A_d^{(i+1)} ad + A_{d-1}^{(i+1)}) + M^{d-1} B \alpha_{i+1}^b (A_d^{(i+1)} bd + A_{d-1}^{(i+1)}) \\ &\quad + M^{d-1} C \alpha_i^c (A_d^{(i)} cd + A_{d-1}^{(i)}) = 0; \end{aligned}$$

and by straightforward calculations we obtain

$$\begin{aligned} A\alpha_i^a + B\alpha_i^b + Cq_i\alpha_{i+1}^c &= 0; \\ A\alpha_{i+1}^a + B\alpha_{i+1}^b + \frac{C}{q_i}\alpha_i^c &= 0; \\ Aa\alpha_i^a + Bb\alpha_i^b + Ccq_i\alpha_{i+1}^c &= 0; \\ Aa\alpha_{i+1}^a + Bb\alpha_{i+1}^b + \frac{C}{q_i}c\alpha_i^c &= 0. \end{aligned} \tag{18}$$

Computing from the first equation $Cq_i\alpha_{i+1}^c$ and inserting into the third equation we get

$$A(a-c)\alpha_i^a + B(b-c)\alpha_i^b = 0.$$

The second and forth equation lead to the same relation for α_{i+1} . Therefore either $\alpha_i = \alpha_{i+1}$ or the sequence is degenerate or $a = c$ and $b = c$, hence in any case we obtain a contradiction. Therefore we have $d = 0$.

Let us investigate the first two equations of (18). By using the fact that $\alpha_{i+1}^n = \alpha_i^{-n}\zeta_i^n$ for all integers n the second equation can be rewritten as

$$A\alpha_i^{-a}\zeta_i^a + B\alpha_i^{-b}\zeta_i^b = -\frac{C\zeta_i^c}{q_i\alpha_{i+1}^c}.$$

Since the right side is obviously not zero we also have $A\alpha_i^{-a}\zeta_i^a + B\alpha_i^{-b}\zeta_i^b \neq 0$ and therefore we can write

$$-\frac{C\zeta_i^c}{A\alpha_i^{-a}\zeta_i^a + B\alpha_i^{-b}\zeta_i^b} = q_i\alpha_{i+1}^c.$$

Inserting into the first equation yields

$$A\alpha_i^a + B\alpha_i^b - \frac{C^2\zeta_i^c}{A\alpha_i^{-a}\zeta_i^a + B\alpha_i^{-b}\zeta_i^b} = 0$$

or in expanded form

$$(A^2\zeta_i^a + B^2\zeta_i^b - C^2\zeta_i^c) + AB\zeta_i^a\alpha_i^{b-a} + AB\zeta_i^b\alpha_i^{a-b} = 0.$$

Note that α_{i+1} satisfies the same equation. Let us assume $a > b$, then α_i and α_{i+1} are of the form $\xi_i\gamma_i^{1/(a-b)}$ or $\xi_{i+1}\gamma_{i+1}^{1/(a-b)}$, where ξ_i, ξ_{i+1} are roots of unity and γ_i and γ_{i+1} are roots of the polynomial

$$X^2 + \frac{A^2\zeta_i^a + B^2\zeta_i^b - C^2\zeta_i^c}{AB\zeta_i^b}X + \zeta_i^{a-b}.$$

Note that α_i and α_{i+1} cannot be both of the form $\xi\gamma_i^{1/(a-b)}$ or $\xi\gamma_{i+1}^{1/(a-b)}$, with ξ some root of unity, since otherwise the recurrence would be degenerate. Inserting for A, B and C the values 1 and -2 according to the cases that may occur we obtain the sequences listed in Table 1.

Before we proceed with the case $f_n f_m f_k = 0$ we want to demonstrate this case by an example: Let us choose $r = 2$ and $\alpha_1 = 2 + \sqrt{5}$, i.e. $\alpha_2 = 2 - \sqrt{5}$. Moreover, we choose $a = 2$ and $b = c = 1$. Therefore all sequences of the form

$$f_n = c_0 \left((2 + \sqrt{5})^n - (2 - \sqrt{5})^n \frac{47 + 21\sqrt{5}}{2} \right)$$

have infinitely many three-term arithmetic progressions. We choose $c_0 = \frac{1 + \frac{21\sqrt{5}-47}{2}}{15}$ and insert for $n = 0$ and $n = 1$ and observe that $2 + \sqrt{5}$ is a root of $X^2 - 4X - 1$. Then we see that the sequence f_n comes from the recurrence

$$f_{n+2} = 4f_{n+1} + f_n, \quad f_0 = -3, \quad f_1 = 2.$$

Therefore f_n is defined over the integers and contains infinitely many three-term arithmetic progressions $(f_{2n+2}, f_{-2n+1}, f_{2n+1})$.

Now let us consider the case $f_m f_n f_k = 0$. As arguing in the previous case we have to deal with the equation $f_n = 2f_k$. By Proposition 2 we either have

$$2p_i(n)\alpha_i^n = p_i(k)\alpha_i^k$$

for each i or

$$2p_i(n)\alpha_i^n = p_{i+1}(k)\alpha_{i+1}^k; \quad 2p_{i+1}(n)\alpha_{i+1}^n = p_i(k)\alpha_i^k$$

for each odd i . The first equation corresponds to the case treated above. Therefore, we may assume $n = Mt + a$ and $k = -Mt + b$. We multiply the first equation by $\alpha_{i+1}^{Mt}/p_i(Mt + a)$ and the second by $\alpha_i^{Mt}/p_{i+1}(Mt + a)$ and take the limit for $t \rightarrow \infty$. Then we get

$$(19) \quad 2\alpha_i^a = \alpha_{i+1}^b q_i, \quad 2\alpha_{i+1}^a = \alpha_{i+1}^b 1/q_i,$$

where $q_i = \lim_{t \rightarrow \infty} p_i(tM + a)/p_{i+1}(-Mt + b)$. Now eliminating q_i from the first equation of (19) yields

$$4\alpha_i^{a-b} = \alpha_{i+1}^{b-a}$$

and multiplying by α_{i+1}^{a-b} yields

$$4\zeta^{a-b} = 1$$

a contradiction.

Note that in the case of $f_m = -f_k$ we obtain by the same computation $\zeta^{a-b} = 1$ which would yield further solutions. Therefore we have excluded these cases in Theorem 1.

3.3. The exceptional case. First, we consider the case $f_m f_n f_k \neq 0$. Then by Proposition 3 we may assume $n = cN^s + \gamma$, $m = cN^s + as + b$ and $k = cN^s + a's + b'$, $p_i(n) = \gamma_i(n - \gamma)$, $N = \alpha_i^{q_i}$ for some rational number q_i and (9) is satisfied. In order to treat several cases at once we assume

$$n = cN^s + as + b, \quad m = cN^s + a's + b', \quad k = cN^s + a''s + b''$$

with $a \geq a' \geq a''$ and one of a, a' and a'' is zero and the corresponding b is equal to γ . Then by Proposition 3 we know that for all $\alpha_i, i = 1, \dots, r$ we have

$$\begin{aligned} A\gamma_i(cN^s + as + b - \gamma)\alpha_i^{cN^s + as + b} + B\gamma_i(cN^s + a's + b' - \gamma)\alpha_i^{cN^s + a's + b'} \\ + C\gamma_i(cN^s + a''s + b'' - \gamma)\alpha_i^{cN^s + a''s + b''} = 0. \end{aligned}$$

For reasons of notation let us drop the indices. Then the equation above can be written as

$$\begin{aligned} (20) \quad & A\alpha^{(q+a)s+b} + B\alpha^{(q+a')s+b'} + C\alpha^{(q+a'')s+b''} \\ & + A(as + b - \gamma)\alpha^{as+b} + B(a's + b' - \gamma)\alpha^{a's+b'} \\ & + C(a''s + b'' - \gamma)\alpha^{a''s+b''} = 0. \end{aligned}$$

Let us assume we have $|\alpha| > 1$ (for all i). Therefore we have $q > 0$. So the maximal coefficient of s in the exponents of (20) is $q + a$. By dividing by $\alpha^{(q+a)s}$ we see that $(q+a)s+b$ cannot be the only maximal exponent. Otherwise every other term than $A\alpha^b$ would converge to 0 and hence $Ac = 0$, a contradiction. So either $q + a = a$ or $q + a = q + a'$. The first case can be excluded since otherwise $q = 0$ and hence $N = 1$. Therefore we have $a = a'$. If a third exponent would be also maximal we would have again either $q = 0$ or $a = a' = a'' = 0$. Now the second case would yield a situation as treated in subsection 3.1. Since the leading terms must cancel, we get

$$A\alpha^{(q+a)s+b} + B\alpha^{(q+a)s+b'} = 0$$

and in particular

$$-\frac{A}{B} = \alpha^{b'-b}.$$

Since α is not a root of unity we must have $AB = -2$, hence $C = 1$. Moreover, since m and n must be both integers also $b - b'$ is an integer, hence $\alpha^K = 2$, with $K = |b - b'| \in \mathbb{Z}$ (remember $|\alpha| > 1$). Of course b and b' depend on the exceptional family e.g. $\mathcal{E}_j^{(k)}$ but not on the root α_i . So for all roots α_i we have the same K , hence there exists only one α since otherwise our recurrence would be degenerate. Moreover, observe that also $A(b - b') > 0$ holds. Using these facts we have

$$B\alpha^{a's+b'} = B\alpha^{as+b+(b'-b)} = -A\alpha^{as+b}$$

and together with (20) we get

$$c\alpha^{(q+a'')s+b''} + A(b - b')\alpha^{as+b} + C(a''s + b'' - \gamma)\alpha^{a''s+b''} = 0.$$

Now the highest exponent is either $as + b$ or $(q + a'')s + b''$, but in any case $a''s + b''$ is smaller (otherwise $q = 0$). Since a single maximum yields a contradiction we deduce similar as above

$$c\alpha^{b''} + A(b - b')\alpha^b = 0,$$

which also implies

$$C(a''s + b'' - \gamma)\alpha^{a''s+b''} = 0,$$

i.e. $a'' = 0$ and $b'' = \gamma$ and so the equation above turns into

$$(21) \quad c + A(b - b')\alpha^{b-\gamma} = 0.$$

From the equation above we also deduce that $b - \gamma \in \mathbb{Z}$ since otherwise $\alpha^{b-\gamma}$ is irrational and so also the left side of the equation, a contradiction. Note that in the case $|\alpha| < 1$ and by assuming $a'' \geq a' \geq a$ we obtain the same conclusions, except

$K = -|b - b'|$ and $A(b - b') < 0$. Let us now assume $n, m, k > 0$ then we have $c > 0$ and additionally let us assume $|\alpha| > 1$ then we also have $K \geq 1 \in \mathbb{Z}$ and by (21) we deduce $\alpha \notin \mathbb{R}^+$ but $\alpha^{b-\gamma} \in \mathbb{Q}^-$ because of (21). But the rational power of a negative rational never can be 2, hence a contradiction. This shows that c and K cannot be both positive.

Now we want to prove that a, a', b, b' and c are integral. From the paragraph above we may assume $N = 2$. Since for all $s \in \mathbb{Z}$ with $s > 0$ the quantities

$$n(s) = c2^s + \gamma, \quad m(s) = c2^s + as + b, \quad k(s) = c2^s + as + b'$$

must be integers and since 2^s and as are periodic modulo each prime $p > 2$ with period dividing $p - 1$ and p respectively and since 2^s is not constant modulo p we deduce that the denominator of c is a power of 2 and therefore $c2^s$ is for large s an integer, which yields that a, a', b, b' and γ are integers (at least for large s and hence for all s). Therefore also c has to be an integer.

Now let us consider the case, where f_n, f_m or f_k vanishes. This leads to an equation of the form (5), but Proposition 3 tells us that such an equation has no additional solutions.

3.4. Proof of Corollary 1. The last subsection of this section is devoted to the proof of Corollary 1. First, we note that since we allow only positive indices the symmetric case is excluded. The exceptional case is also excluded since α with $\alpha^K = 2$ has to be an (algebraic integer), hence $|\alpha| > 1$. But in this case we have $K \geq 1$, hence by Theorem 1 we have $c < 0$, contradicting the fact $n, k, m > 0$. So only the general case remains. But Lemma 3 below will show that $2X^a - X^b - 1 = (X^d - 1)g(X)$ with $g(X)$ irreducible and $d = \gcd(a, b)$. Hence $P(X) = g(X)$ but has no integral roots, hence $f_n \notin \mathbb{Z}$ if n is large. A similar argument applies to the case $X^a + X^b - 2$, but here we conclude $f_n \notin \mathbb{Z}$ if n tends to $-\infty$.

4. THE BINARY CASE

4.1. Exceptional case. Since f_n is defined over the rationals and $\alpha = 2^K$ we have $K = \pm 1$, i.e. $f_n = R(n - \gamma)2^{Kn}$, where $R \in \mathbb{Q}^*$. Therefore we have $q = a = a' = K$ and without loss of generality we may assume $A = 1$. Then we have $b - b' = K$ and $c + A(b - b')\alpha^{b-\gamma} = c + K2^{K(b-\gamma)} = 0$. Moreover we have

$$\begin{aligned} m &= c2^s + as + b = -K2^{s+Kb-K\gamma} + Ks + b; \\ n &= c2^s + a's + b' = -K2^{s+Kb-K\gamma} + Ks + b - K; \\ k &= c2^s + \gamma = -K2^{s+Kb-K\gamma} + \gamma. \end{aligned}$$

In particular substituting s for $s + Kb$ we see that (f_m, f_n, f_k) is an three-term arithmetic progression if

$$m = -K2^{s-K\gamma} + Ks, \quad n = -K2^{s-K\gamma} + Ks - K, \quad k = -K2^{s-K\gamma} + \gamma.$$

For $s > K\gamma + \frac{1+K}{2}$ these are distinct integers. Substituting 1 and -1 for K we get the statement for the exceptional case.

4.2. Symmetric case. Let us now consider the symmetric case. We keep the notation of the previous section. Since $f_n = c_1\alpha_1^n + c_2\alpha_2^n$ is defined over the rationals we have $\alpha_1\alpha_2 = \pm 1$. In the case of $\alpha_1\alpha_2 = 1$ the first two equations

of (18) yield $\alpha_1 = \alpha_2 = q = 1$, which is excluded. Therefore we have $\alpha_1\alpha_2 = -1$. The case $C = -2$ yields polynomials of the form

$$X^2 + 4X - 1, \quad X^2 - 4X - 1, \quad X^4 + 6X^2 + 1, \quad X^4 - 2X^2 + 1$$

and in the case of $B = -2$ or $A = -2$ we obtain the polynomials

$$X^2 - 2X + 1, \quad X^2 + X - 1, \quad X^2 - X - 1, \quad X^2 + 2X - 1, \quad X^2 - 2X - 1, \quad X^4 - 3X^2 + 1,$$

where X is of the form x^{a-b} or x^{b-a} depending on the sign of $a - b$. Since α_1, α_2 have to be quadratic integers not roots of unity, the only possibilities are in the case of $C = -2$

$$\alpha_1 = \pm 2 \pm \sqrt{5} \quad \text{and} \quad \alpha_1 = \frac{\pm 1 \pm \sqrt{5}}{2}$$

and in the case of $B = -2$ or $A = -2$

$$\alpha_1 = \pm 1 \pm \sqrt{2} \quad \text{and} \quad \alpha_1 = \frac{\pm 1 \pm \sqrt{5}}{2}.$$

Note that except $\pm 2 \pm \sqrt{5}$ all of these are fundamental units and we have $\pm 2 \pm \sqrt{5} = \left(\frac{\pm 1 \pm \sqrt{5}}{2}\right)^3$ choosing the signs appropriately. In particular for all these integers we have to choose a and b such that $|a - b| = 1$ or $|a - b| = 3$. The last case may only occur for $\alpha_1 = \frac{\pm 1 \pm \sqrt{5}}{2}$ and $C = -2$.

Let us consider the case $C = -2$ in more detail. In this case we may assume without loss of generality $a > b$. We know that

$$q = \frac{\alpha_1^{a+c} + \alpha_1^{b+c}}{2}(-1)^c$$

and therefore

$$f_0 = C_0 \left(1 + \frac{\alpha_1^{a+c} + \alpha_1^{b+c}}{2}(-1)^c \right)$$

and

$$f_1 = C_0 \left(\alpha_1 + \frac{\alpha_1^{a+c-1} + \alpha_1^{b+c-1}}{2}(-1)^{c+1} \right).$$

If f_n is defined over the rationals then for a C_0 such that f_0 is rational also f_1 has to be rational. If we choose $C_0 = 1 + \frac{\alpha_2^{a+c} + \alpha_2^{b+c}}{2}(-1)^c$ we certainly have $f_0 \in \mathbb{Q}$ since this is the norm of C_0 . Hence we have to consider f_1 :

$$\begin{aligned} f_1 &= \left(1 + \frac{\alpha_2^{a+c} + \alpha_2^{b+c}}{2}(-1)^c \right) \left(\alpha_1 + \frac{\alpha_1^{a+c-1} + \alpha_1^{b+c-1}}{2}(-1)^{c+1} \right) \\ &= \alpha_1 + \overbrace{\frac{\alpha_2^{a+c-1} + \alpha_2^{b+c-1}}{2}(-1)^{c+1} + \frac{\alpha_1^{a+c-1} + \alpha_1^{b+c-1}}{2}(-1)^{c+1}}^R \\ &\quad - \underbrace{\frac{(\alpha_2^{a+c} + \alpha_2^{b+c})(\alpha_1^{a+c-1} + \alpha_1^{b+c-1})}{4}}_Q. \end{aligned}$$

We note that R is a rational number since it is the trace of an algebraic number. The numerator of Q turns into

$$\alpha_2((-1)^{a+c-1} + (-1)^{b+c-1}) + (-1)^{b+c}\alpha_1^{a-b-1} + (-1)^{b+c-1}\alpha_2^{a-b+1} = (-1)^{b+c}(\alpha_1^{a-b-1} - \alpha_2^{a-b+1}).$$

In the case of $a - b = 1$ we obtain

$$\alpha_1 - \frac{(-1)^{b+c}(1 - \alpha_2^2)}{4}$$

is rational. If we try all possibilities we see that this is possible if and only if $\alpha_1 = 2 \pm \sqrt{5}$ and $b + c \equiv 0 \pmod{2}$ or $\alpha_1 = -2 \pm \sqrt{5}$ and $b + c \equiv 1 \pmod{2}$. In the case of $a - b = 3$ we deduce that

$$\alpha_1 - \frac{(-1)^{b+c}(\alpha_1^2 - \alpha_2^4)}{4}$$

is rational. Note $a - b = 3$ is only possible if $\alpha_1 = \frac{\pm 1 \pm \sqrt{5}}{2}$. Therefore we see that $\alpha_1 = \frac{1 \pm \sqrt{5}}{2}$ if $b + c \equiv 0 \pmod{2}$ and $\alpha_1 = \frac{-1 \pm \sqrt{5}}{2}$ otherwise.

Now, let us consider the case $A = -2$. We may assume $a - b = 1$ (note $b - a = 1$ yields the same computations and the same results as case $B = -2$ and $a - b = 1$ treated below) we have

$$q = (-1)^c(\alpha_1^{b+c} - 2\alpha_1^{a+c})$$

and therefore we have

$$f_0 = C_0(1 + (-1)^c(\alpha_1^{b+c} - 2\alpha_1^{a+c}))$$

and

$$f_1 = C_0(\alpha_1 + (-1)^{c+1}(\alpha_1^{b+c-1} - 2\alpha_1^{a+c-1})).$$

If we choose $C_0 = 1 + (-1)^c(\alpha_2^{b+c} - 2\alpha_2^{a+c})$ then f_1 has to be a rational. Therefore let us compute

$$\begin{aligned} f_1 &= (1 + (\alpha_2^{b+c} - 2\alpha_2^{a+c})(-1)^c)(\alpha_1 + (\alpha_1^{b+c-1} - 2\alpha_1^{a+c-1})(-1)^{c+1}) \\ &= \alpha_1 + \overbrace{(\alpha_2^{b+c-1} + \alpha_2^{b+c-1})(-1)^{c+1} + (\alpha_1^{b+c-1} - 2\alpha_1^{a+c-1})(-1)^{c+1}}^R \\ &\quad - \overbrace{(\alpha_2^{b+c} - 2\alpha_2^{a+c})(\alpha_1^{b+c-1} - 2\alpha_1^{a+c-1})}^Q \end{aligned}$$

obviously R is rational since it is the trace of an algebraic number. Let us consider Q under the assumption $a - b = 1$:

$$\begin{aligned} &(\alpha_2^{b+c} - 2\alpha_2^{a+c})(\alpha_1^{b+c-1} - 2\alpha_1^{a+c-1}) \\ &= (-1)^{b+c-1}\alpha_2 + 4\alpha_2(-1)^{a+c+1} - 2(-1)^{b+c-1}\alpha_2^{a-b+1} - 2(-1)^{a+c-1}\alpha_2^{b-a+1} \\ &= (-1)^{b+c}(3\alpha_2 + 2\alpha_2^2 - 2). \end{aligned}$$

Therefore

$$\alpha_1 - (-1)^{b+c}(3\alpha_2 + 2\alpha_2^2 - 2)$$

has to be rational. Inserting the possibilities for α_1 and α_2 we see that $\alpha_1 = -1 - \sqrt{2}$ if $b + c \equiv 0 \pmod{2}$ and $\alpha_1 = -\frac{1+\sqrt{5}}{2}$ otherwise.

In the case $B = -2$ we obtain by a similar computation $\alpha_1 = -1 - \sqrt{2}$ if $b + c \equiv 1 \pmod{2}$ and $\alpha_1 = -\frac{1+\sqrt{5}}{2}$ otherwise.

Inserting all possibilities we obtain exactly the sequences listed in Table 1.

4.3. One parameter family. We may exclude the polynomial $2X^a - X^b - 1$ from our considerations since the transformation $X \rightarrow 1/X$ yields the polynomial $-Y^a - Y^b + 2 = (-1)(Y^a + Y^b - 2)$ with $Y = 1/X$ and this equivalent to a transformation $f_n \rightarrow f_{-n}$. Now we have to consider which quadratic polynomials with no roots of unity in their set of roots divide $X^a - 2X^b + 1$ or $X^a + X^b - 2$. The first polynomial was studied by Schinzel [13]:

Lemma 2 (Schinzel). *The polynomial*

$$\frac{X^n - 2X^m + 1}{X^{\gcd(n,m)} - 1}$$

is irreducible over \mathbb{Q} for all $n > m > 0$, except $n = 7k$ and $m = 5k$ or $m = 2k$. Then the polynomial factors into

$$(X^{3k} + X^{2k} - 1)(X^{3k} + X^k - 1)$$

or

$$(X^{3k} + X^{2k} + 1)(X^{3k} - X^k - 1)$$

Therefore we either have $a = 3$ which yields polynomials listed in Table 2 or $a - \gcd(a, b) = 2$ and $a > 3$. Since $a - \gcd(a, b) \geq a/2$ if $a > b$ we deduce $a = 4$ and $b = 2$ but $X^4 - 2X^2 + 1 = (X^2 - 1)^2$ which yields only degenerate or unitary sequences.

We consider now the case $X^a + X^b - 2$. Schinzel [14] proves in particular that the polynomial $\frac{X^a + X^b - 2}{X^{\gcd(a,b)} - 1}$ is irreducible if $a/\gcd(a, b) < C$ where C is an absolute computable constant. However the constant is by far too large ($> (10^8!)^4$) to prove the irreducibility of $X^a + X^b - 2$. Also the bound in [15] is too large ($\sim 10^8$). But following ideas of Schinzel [13] and Ljunggren [9] we can show

Lemma 3. *The polynomial*

$$\frac{X^n + X^m - 2}{X^{\gcd(n,m)} - 1}$$

is irreducible over \mathbb{Q} for all $n > m > 0$.

In order not to interrupt the proof of Theorem 2 we postpone the proof of the lemma to the next subsection.

By Lemma 3 we have $a = 3$ which only yields polynomials listed in Table 2, or we have $a = 4$ and $b = 2$. But

$$X^4 + X^2 - 2 = (X - 1)(X + 1)(X^2 - 2)$$

and hence each factor yields a degenerate or unitary recurrence. Therefore we have proved Theorem 2 completely apart from Lemma 3.

4.4. Proof of Lemma 3. Suppose that $f(X)g(X) = X^n + X^m - 2$. Therefore we have $f(0)g(0) = -2$ and without loss of generality we have $|f(0)| = 1$. On the other hand a root of $X^m + X^n - 2$ cannot have a root α such that $|\alpha| < 1$ since otherwise $|\alpha^n + \alpha^m - 2| \geq 2 - |\alpha|^n - |\alpha|^m > 0$. Therefore all roots of $X^m + X^n - 2$ have absolute value at least 1. Since the product of all roots of $f(X)$ has absolute value 1 no root α has absolute value greater than 1, since then another root α' must have absolute value less than 1, i.e. all roots α of $f(X)$ satisfy $|\alpha| = 1$. Therefore we

can write $\alpha = e^{2\pi i y}$ and since α is a root of $X^m + X^n - 2$ considering real parts we obtain $\cos(2\pi m y) = \cos(2\pi n y) = 1$, hence $\sin(2\pi m y) = \sin(2\pi n y) = 0$. Therefore $2y$ is rational and its denominator divides n and m . Therefore α is an $\gcd(n, m)$ -th root of unity, i.e. $f(X) | X^{\gcd(n, m)} - 1$. ■

4.5. Proof of Corollaries 2 and 3. By Theorem 1 and Theorem 2 we know that the companion polynomial must be one of the polynomials listed in Table 2. The root α with maximal absolute value of all polynomials listed in Table 2 except the polynomial $X^2 - X - 1$ is either not real or satisfies $\alpha < 1$. Therefore sequences with such companion polynomial are not increasing. Therefore a sequence satisfying the conditions of Corollary 2 has companion polynomial $X^2 - X - 1$. Since a binary recurrence is uniquely determined by its companion polynomial and the values at f_0 and f_1 the first part of Corollary 2 is proved.

It is well known (Binet's formula) that the Fibonacci sequence is defined by the explicit formula

$$f_n = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right).$$

Since f_n is increasing a non-trivial three-term arithmetic progression with $f_m > f_n > f_k$ fulfills $m > n > k \geq 0$ and in particular for $k > 0$ we have

$$\begin{aligned} \left(\frac{1+\sqrt{5}}{2} \right)^m + \left(\frac{1+\sqrt{5}}{2} \right)^k - 2 \left(\frac{1+\sqrt{5}}{2} \right)^n = \\ \left(\frac{1-\sqrt{5}}{2} \right)^m + \left(\frac{1-\sqrt{5}}{2} \right)^k - 2 \left(\frac{1-\sqrt{5}}{2} \right)^n \end{aligned}$$

Dividing this equation by $\left(\frac{1+\sqrt{5}}{2} \right)^k$ and using some simple estimations we obtain

$$(22) \quad \left| 2 \left(\frac{1+\sqrt{5}}{2} \right)^{n-k} - \left(\frac{1+\sqrt{5}}{2} \right)^{m-k} - 1 \right| < 4 \left(\frac{3-\sqrt{5}}{2} \right)^k.$$

Let us put $m - k = a$ and $n - k = b$ in (22), then we obtain

$$\left| \left(\frac{1+\sqrt{5}}{2} \right)^b \left(\left(\frac{1+\sqrt{5}}{2} \right)^{a-b} - 2 \right) + 1 \right| < 4 \left(\frac{3-\sqrt{5}}{2} \right)^k.$$

First, let us consider the case $a - b = 1$. We may exclude $b = 2$ since this yields the known family of three-term arithmetic progressions. With $a - b = 1$ we obtain

$$\left| 1 - \left(\frac{1+\sqrt{5}}{2} \right)^{b-2} \right| < 4 \left(\frac{3-\sqrt{5}}{2} \right)^k.$$

For $b = 1$ and $b = 3$ we obtain $k \leq 2$ and for $b > 3$ we have $k = 0$ a contradiction. Therefore the only possibilities for the triple (m, n, k) are:

$$(3, 2, 1), (4, 3, 2), (5, 4, 1), \text{ and } (6, 5, 2).$$

The only new triple that indeed provides an arithmetic progression is $(4, 3, 2)$, but this triple is listed in the corollary. For $a - b = 2$ we obtain the inequality

$$\left| 1 - \left(\frac{1 + \sqrt{5}}{2} \right)^{b-1} \right| < 4 \left(\frac{3 - \sqrt{5}}{2} \right)^k.$$

For $b = 2$ we obtain $k \leq 2$ and for $b > 2$ we have $k = 0$, a contradiction. In the case $b = 1$ the original equation turns into

$$\begin{aligned} \left(\frac{1 + \sqrt{5}}{2} \right)^{k+3} + \left(\frac{1 + \sqrt{5}}{2} \right)^k - 2 \left(\frac{1 + \sqrt{5}}{2} \right)^{k+1} = \\ \left(\frac{1 - \sqrt{5}}{2} \right)^{k+3} + \left(\frac{1 - \sqrt{5}}{2} \right)^k - 2 \left(\frac{1 - \sqrt{5}}{2} \right)^{k+1}. \end{aligned}$$

Dividing through $\left(\frac{1 + \sqrt{5}}{2} \right)^k$ and a simple estimation on the right side yields

$$2 = \left(\frac{1 + \sqrt{5}}{2} \right)^3 + 1 - 2 \cdot \frac{1 + \sqrt{5}}{2} < 4 \left| \frac{1 - \sqrt{5}}{2} \right|^{2k}.$$

Therefore we have $k = 0$, also a contradiction. Therefore the triple (m, n, k) is either $(5, 3, 1)$ or $(6, 4, 2)$. But both do not yield arithmetic progressions. In the case of $a - b \geq 3$ the left side of (22) is at least $\frac{7 + \sqrt{5}}{2} > 4$ which is larger than the right side, hence this case does not occur.

Now we consider the case $k = 0$. Then inequality (22) turns into

$$\left| \left(\frac{1 + \sqrt{5}}{2} \right)^{m-n} - 2 \right| < 3 \left(\frac{1 - \sqrt{5}}{2} \right)^{2n}.$$

If $m - n = 1$ we obtain

$$\left| \frac{3 - \sqrt{5}}{2} \right| < 3 \left(\frac{3 - \sqrt{5}}{2} \right)^n,$$

i.e. we have $n = 1, 2$. For $m - n = 2$ we have

$$\left| \frac{1 - \sqrt{5}}{2} \right| < 3 \left(\frac{3 - \sqrt{5}}{2} \right)^n,$$

i.e. we have $n = 1$. For $m - n > 2$ we have

$$\sqrt{5} \leq \left| \left(\frac{1 + \sqrt{5}}{2} \right)^{m-n} - 2 \right| < 3 \left(\frac{3 - \sqrt{5}}{2} \right)^n,$$

i.e. $n = 0$, a contradiction. Therefore we have for the triple (m, n, k) only the possibilities $(2, 1, 0)$, $(3, 2, 0)$ or $(3, 1, 0)$ which all appear in the statement of the corollary or yield no arithmetic progression.

We have seen that an arithmetic three-term progression in the Fibonacci sequence comes from the infinite family or has highest index at most 4. Therefore an arithmetic four-term progression might have highest index at most 5. Writing

down f_n for $n = 0, \dots, 5$ we see that the only four-term arithmetic progressions appearing are those written down in the corollary.

Now we turn to the proof of Corollary 3. A unitary increasing binary recurrence defined over the rationals is of the form $f_n = ca^n + d(\pm 1)^n$, where $1 < a \in \mathbb{Q}$ and $c, d \in \mathbb{Q}$. In the case of $+1$ also the sequence a^n yields infinitely many three-term arithmetic progression. But in this case we have $a^m - 2a^n + a^k = 0$ or $a^x - 2a^y + 1 = 0$, with $x > y > 0$. On the other hand we claim $a^x - 2a^y + 1 \neq 0$ if $a \neq \pm 1$, but $a = \pm 1$ yields f_n degenerate or constant. For the proof of the claim see at the end of this section. Therefore we assume the -1 case. Note that the equation $2f_n = f_m + f_k$ with $m > n > k \geq 0$ turns into

$$(23) \quad a^m - 2a^n + a^k = -\frac{d}{c}((-1)^m - 2(-1)^n + (-1)^k) < C_0$$

where C_0 is an absolute constant. If $a > 2$ the left side tends to ∞ whenever $m \rightarrow \infty$. Therefore let us consider the case $2 > a > 1$. If $a^k < C_0$ the inequality above turns into $|a^m - 2a^n| < 2C_0$ and in the case of $a^k \geq C_0$ we divide by a^k and obtain $|a^{m-k} - 2a^{n-k}| < 1 + C_0/a^k \leq 2$. In any case there is a constant C_1 such that

$$|a^y - 2a^x| < C_1,$$

where $y = m, m - k$ and $x = n, n - k$ depending on the size of a^k . By a standard application of Baker's theory of linear forms in logarithms (see e.g. [1]) the inequality has only finitely many solutions x and y . Note that a and 2 are multiplicatively independent. For small k we therefore deduce that only finitely many solutions to (23) exist. We claim that $a^x - 2a^y + 1 \neq 0$ for $2 > a > 1$ with $a \in \mathbb{Q}$. If this is true, then since $m - k$ and $n - k$ obtain only finitely many values we find a constant C_2 such that

$$C_0 < a^m - 2a^n + a^k = (a^{m-k} - 2a^{n-k} + 1)a^k < C_2a^k.$$

Hence k takes only finitely many values, i.e. (23) has only finitely many solutions. Therefore we deduce $a = 2$ and $f_n = c2^n + d(-1)^n$. Since $f_0 = 0$ and $f_1 = 1$ we deduce $f_n = \frac{2^n - (-1)^n}{3}$ which fulfills all requirements of the corollary.

So we are left to prove the claim $a^x - 2a^y + 1 \neq 0$ for $2 > a > 1$ with $a \in \mathbb{Q}$. But by Lemma 2 we know that the only linear factors of the polynomial $X^n - 2X^m + 1$ may only have the linear factors $(X - 1)$ and $(X + 1)$.

5. THE TERNARY CASE

To obtain infinite families in the symmetric case or in the exceptional case the degree of the companion polynomial has to be even an obvious contradiction. So only the case of one parameter families remains.

First, let us consider factors of degree 3 of $X^a - 2X^b + 1$. Due to Schinzel's result (see Lemma 2) we may assume $a = 4, 5, 6, 7$. In case of $a = 4$ we may choose $b = 1$ or $b = 3$ (for $b = 2$ the non-cyclotomic factor would be of degree ≤ 2). The corresponding irreducible non-cyclotomic factors of degree 3 are

$$X^3 - X^2 - X - 1 \quad \text{and} \quad X^3 + X^2 + X - 1.$$

If $a = 5$ we have no factor of degree 3. In the case of $a = 6$ we may choose $b = 3$ but then we have $X^6 - 2X^3 + 1 = (X^3 - 1)^2$ which has only cyclotomic factors. In the case of $a = 7$ we may choose $b = 2$ or $b = 5$ and the non-cyclotomic factors are for $b = 2$

$$X^3 + X^2 + 1 \quad \text{and} \quad X^3 - X - 1$$

and for $b = 5$

$$X^3 + X^2 - 1 \text{ and } X^3 + X - 1.$$

So all polynomials listed in Theorem 3 were found and there are no further possibilities left.

Now we consider factors of degree 3 of $X^a + X^b - 2$. Due to Lemma 3 we know that $a = 4, 5, 6$. Similar as above we may exclude $a = 5$. In the case of $a = 6$ the non-cyclotomic factor of degree 3 is $X^3 - 2$ but this polynomial only yields degenerate recurrences. So we are left to the case $a = 4$ and $b = 1$ or $b = 3$. Therefore we obtain the polynomials

$$X^3 + X^2 + X + 2 \text{ and } X^3 + 2X^2 + 2X + 2.$$

By replacing X by $1/X$ and expanding, the case $2X^a - X^b - 1$ is equivalent to the case $X^a + X^b - 2$. Therefore we obtain 2 further polynomials namely

$$2X^3 + X^2 + X + 1 \text{ and } 2X^3 + 2X^2 + 2X + 1.$$

■

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Á. PINTÉR
INSTITUTE OF MATHEMATICS
UNIVERSITY OF DEBRECEN
H-4010 DEBRECEN, HUNGARY
E-mail address: `apinter@math.unideb.hu`

V. ZIEGLER
INSTITUTE FOR ANALYSIS AND
COMPUTATIONAL NUMBER THEORY,
GRAZ UNIVERSITY OF TECHNOLOGY
STEYRERGASSE 30/IV,
A-8010 GRAZ, AUSTRIA
E-mail address: `ziegler@finanz.math.tugraz.at`